On conjugacy in regular epigroups

Ganna Kudryavtseva

Department of Mechanics and Mathematics,
Kyiv Taras Shevchenko University,
Volodymyrs'ka, 60, Kuiv, 01033, Ukraine
akudr@univ.kiev.ua

Abstract

Let S be a semigroup. The elements $a, b \in S$ are called primarily conjugate if $a = xy$ and $b = yx$ for certain $x, y \in S$. The relation of conjugacy is defined as the transitive closure of the relation of primary conjugacy. In the case when S is a monoid, denote by G the group of units of S. Then the relation of G-conjugacy is defined by $a \sim_G b \iff a = g^{-1}bg$ for certain $g \in G$. We establish the structure of conjugacy classes for regular epigroups (i.e. semigroups such that some power of each element lies in a subgroup). As a corollary we obtain a criterion of conjugacy in terms of G-conjugacy for factorizable inverse epigroups. We show that our general conjugacy criteria easily lead to known and new conjugacy criteria for some specific semigroups, among which are the full transformation semigroup and the full inverse symmetric semigroup over a finite set, the linear analogues of these semigroups and the semigroup of finitary partial automatic transformations over a finite alphabet.

AMS 2000 Subject Classification 20M10, 20M20.

1   Introduction

The notion of conjugacy in semigroups can be generalized from the corresponding notion for groups in several ways. Perhaps, the two most natural and commonly used notions are the relations $\sim_G$ and $\sim$, whose definitions below are taken from [8].

Let S be a monoid and G the group of units of S. The relation $\sim_G$, called *G-conjugacy*, is defined as $a \sim_G b$ if and only if $a = g^{-1}bg$ for certain

$g \in G$. Let now S be a semigroup. We call the elements $a, b \in S$ *primarily S-conjugate* if there exist $x, y \in S$ such that $a = xy$, $b = yx$. This will be denoted by $\sim_{pS}$ or just by $\sim_p$ when this does not lead to ambiguity. The relation $\sim_p$ is reflexive and symmetric while not transitive in the general case. Denote by $\sim$ the transitive closure of the relation $\sim_p$. If $a \sim b$ then a and b are said to be *S-conjugate* or just *conjugate*. It is easy to see that in the case of group, both $\sim_G$ and $\sim$ coincide with usual group conjugacy. Besides, for a monoid S there is an inclusion $\sim_G \subset \sim$.

The structure of conjugacy and G-conjugacy classes for some specific regular semigroups was studied in a number of papers ([2], [6], [7], [1], [10]), see also the monograph [9]. For the relation of conjugacy the structure of conjugacy classes usually happens to be more complicated than for the relation of G-conjugacy.

The present paper is devoted to the systematic study of the conjugacy relation in regular epigroups (an *epigroup* or a *group-bound semigroup* is a semigroup such that some power of each its element lies in a subgroup) and is organized as follows. In the Preliminaries we collect some notation used throughout the paper and cite some well-known facts about the structure of $\mathcal{D}$-classes. In Section 3 we establish a criterion of conjugacy of two group elements of a given semigroup. In section 4 we establish Theorem 2, which gives a criterion of conjugacy of two group-bound elements of a regular semigroup and Theorem 3, which provides a criterion of conjugacy in terms of G-conjugacy for factorizable inverse epigroups. In Section 5 we show that conjugacy criteria for many important specific examples of regular epigroups can be derived in a unified way from our main results. In particular, we give short and very clear proofs for some known conjugacy criteria. Besides, for the first time we formulate and prove conjugacy criteria for the semigroups $\mathrm{PAut}(V)$, $\mathrm{PEnd}(V)$, $Fin\mathcal{ISA}(X)$, $Fin\mathcal{A}(X)$ and $Fin\mathcal{PA}(X)$ (see the explanations for the notations at the appropriate places of the paper).

In the case when a regular semigroup S is not group-bound, the problem of description of conjugacy classes of S seems to be much more complicated, in particular, even for such a classical semigroup as $\mathcal{T}(\mathbb{N})$ the conjugacy classes are not classified yet. At the same time, for the semigroup $\mathcal{IS}(\mathbb{N})$, which is also not an epigroup, the conjugacy classes are described (see [6]).

In Appendix A we show that despite finiteness of X the semigroup $\mathcal{ISA}(X)$ of partial automatic permutations over a finite alphabet X ($|X| \geq 2$) is not an epigroup. This implies, in particular, that in $\mathcal{ISA}(X)$ there are conjugacy classes without group elements showing that the conjugacy criterion for the semigroup $\mathcal{ISA}(X)$ announced in Theorems 3,4 of [10] fails to give sufficient condition of conjugacy. At the same time we have substantial arguments that the description of conjugacy classes in $\mathcal{ISA}(X)$ can be obtained using the

methods from [3, 6]. A paper devoted to this question is now in preparation.

2 Preliminaries

Let S be a semigroup and $a \in S$. The class containing a with respect to the $\mathcal{H}$- ($\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{D}$-, $\mathcal{J}$-) Green's relations will be denoted by H_a (L_a, R_a, D_a, J_a). The following well-known facts about the structure of $\mathcal{D}$-classes will be used and referred to in the sequel. Their proofs can be found, for example, in [4].

Proposition 1 (see [4], Theorem 1.2.5, p. 18). *Let $a, b \in S$. Then $ab \in R_a \cap L_b$ if and only if $R_b \cap L_a$ contains an idempotent. In particular, a triple a, b, ab belongs to the same $\mathcal{H}-$ class if and only if this $\mathcal{H}$-class is a group.*

Proposition 2 (see [4], Theorems 1.2.7 and 1.2.8., pp. 18, 19). *Let $e, f \in S$ be idempotents and $e \mathcal{D} f$. Then for any $t \in R_e \cap L_f$ there is an inverse t' of t such that $t' \in R_f \cap L_e$. Furthermore, the maps $\rho_t \circ \lambda_{t'} : H_e \to H_f$ and $\rho_{t'} \circ \lambda_t : H_f \to H_e$ defined via $x \mapsto t'xt$ and $x \mapsto txt'$, respectively, are mutually inverse isomorphisms.*

Recall that an element $a \in S$ is said to be a *group element* provided a belongs to a certain subgroup of S. It is easily seen and well-known that for a group element $a \in S$ its $\mathcal{H}$-class H_a is a group (in fact, H_a is a maximal subgroup of S). Denote by a^{-1} the (group) inverse of a in H_a.

Let $a \in S$ and there exists $t \in \mathbb{N}$ such that a^t is a group element. In this case a is called a *group-bound element*. S is called an *epigroup* (or a *group-bound semigroup*) provided that each element of S is group-bound.

The following fact is known and is easily proved.

Lemma 1. *The following statements are equivalent.*

1. *$a^k \mathcal{H} a^t$ for some $k > t$.*

2. *$a^i \mathcal{H} a^t$ for all $i \geq t$.*

3. *H_{a^t} is a group.*

Let $a \in S$ be a group-bound element and $t \in \mathbb{N}$ is such that H_{a^t} is a group. It follows from Lemma 1 that we can correctly define e_a (the notation goes from [11]) to be the identity element of the group H_{a^t}. Using Lemma 1 one can easily obtain the following (known) useful statement.

Corollary 1. *Suppose a is a group-bound element of S. Then $e_a a = a e_a$ and $a e_a \mathcal{H} e_a$. In particular, $a e_a$ is a group element.*

3   Conjugacy criterion for group elements

We start from conjugacy criterion for group elements of an arbitrary semigroup S generalizing a similar result obtained earlier in [6] for the case when the semigroup S is finite.

Recall that elements $a, b \in S$ are said to be *mutually inverse* provided that $a = aba$ and $b = bab$.

Theorem 1. *Let S be a semigroup and $a, b \in S$ group elements. Then*

1. *$a \sim_p b$ if and only if there exists a pair of mutually inverse elements $u, v \in S$ such that $b = uav$ and $a = vbu$.*

2. *$a \sim b$ if and only if $a \sim_p b$.*

To prove this theorem we will need the following five lemmas.

Lemma 2. *$a \sim_p b$ implies $a^n \sim_p b^n$ for each $n \geq 1$.*

Proof. It is enough to note that if $a = xy$ and $b = yx$ then $a^n = x(yx)^{n-1} \cdot y$ and $b = y \cdot x(yx)^{n-1}$, $n \geq 2$. $\qquad\square$

Lemma 3. *Suppose $a, b \in S$ and $a \sim_p b$. If b belongs to a group, then so does a^2.*

Proof. Assume $a = ts$, $b = st$ for certain $t, s \in S$. That $b = e_b d e_b = e_b s t e_b$ implies $b \in e_b s S^1$. Besides, $e_b s = b b^{-1} e_b s \in b S^1$, whence $b \mathcal{R} e_b s$. Analogously one shows that $b \mathcal{L} t e_b$. Therefore, $e_b s \cdot t e_b \in R_{e_b s} \cap L_{t e_b}$ and thus in view of Proposition 1 $L_{e_b s} \cap R_{t e_b}$ contains an idempotent. Now, since $L_{t e_b} \cap R_{e_b s} = H_b$ is a group, it follows that $t e_b \cdot e_b s = t e_b s \in R_{t e_b} \cap L_{e_b s}$, so that $t e_b s$ is a group element. This implies $(t e_b s)^2 \mathcal{H} t e_b s$. Therefore, since

$$a^2 = tsts = tbs = t e_b d e_b s = (t e_b s)^2,$$

we have that a^2 is a group element. $\qquad\square$

Say that $c, d \in S$ are *conjugate in at most k steps* provided that there are $k \geq 1$ and $c = c_0, c_1 \ldots c_k = d$ such that $c_i \sim_p c_{i+1}$, $0 \leq i \leq k - 1$.

Lemma 4. *Suppose $a, b \in S$ and $a \sim b$. If b belongs to a group, then so does some power of a.*

Proof. Let $k \geq 1$ be such that a and b are conjugate in at most k steps. Show that a^{2^k} is a group element. Apply induction on k. For $k = 1$ the statement follows from Lemma 3.

4

Assume now that $m \geq 1$ and the statement is proved for $k = m$. Let $k = m + 1$. Fix $a = c_0, c_1, \ldots, c_k = b$ such that $c_i \sim_p c_{i+1}$, $0 \leq i \leq k - 1$. Since $c_{k-1} \sim_p b$ then c_{k-1}^2 is a group element by Lemma 3. Beside this, $a^2 \sim_p c_1^2 \sim_p \cdots \sim_p c_{k-1}^2$ due to Lemma 2, which means that a^2 and c_{k-1}^2 are conjugate in at most $k - 1 = m$ steps. Applying the inductive hypothesis we obtain that $(a^2)^{2^{k-1}} = a^{2^k}$ is a group element, as required. $\square$

Corollary 2. *Suppose $a, b \in S$ and $a \sim b$. If b is group-bound then so is a.*

Proof. The statement follows from Lemma 2 and Lemma 4. $\square$

Lemma 5. *Suppose that $a, b \in S$ are group elements and $a \sim_p b$. Then there exist $x, y \in S$ such that $a = xy$, $b = yx$ and $x \in R_a \cap L_b$, $y \in R_b \cap L_a$.*

Proof. Since $a \sim_p b$ then $a = st$ and $b = ts$ for certain $s, t \in S$. It follows that $te_a \mathcal{L} a \mathcal{R} e_a s$. Then $e_a s \cdot te_a \in R_{e_a s} \cap L_{te_a}$ which implies that $L_{e_a s} \cap R_{te_a}$ contains an idempotent by Proposition 1. Since $L_{te_a} \cap R_{e_a s} = H_a$ is a group then $te_a \cdot e_a s = te_a s \in R_{te_a} \cap L_{e_a s}$, so that $te_a s$ is a group element. This implies $(te_a s)^2 \mathcal{H} te_a s$. Therefore, in view of

$$(te_a s)^2 = t \cdot e_a s t e_a \cdot s = tsts = b^2 \mathcal{H} b,$$

we get $te_a s \mathcal{H} b$. Hence $te_a s \mathcal{R} te_a s \mathcal{H} e_b$, whence $e_b te_a = te_a$. Analogously, $se_b t \mathcal{H} a$ and $e_a se_b = e_a s$. But then

$$a = e_a s \cdot te_a = e_a se_b \cdot e_b te_a = e_a \cdot se_b t \cdot e_a = se_b t$$

and analogously $b = te_a s$. Set $x = e_a se_b$, $y = e_b te_a$. We obtain $a = xy$, $b = yx$ and $x \in R_a \cap L_b$, $y \in R_b \cap L_a$ as required. $\square$

Lemma 6. *Let $a, b \in S$ be two group-bound elements. Then $a \sim_p b$ implies $ae_a \sim_p be_b$.*

Proof. Let $n \in \mathbb{N}$ be chosen such that a^n and b^n are group elements. Fix $x, y \in S$ such that $a = xy$ and $b = yx$. Then

$$a^{n+1} = x(yx)^n y = xb^n y = xb^n e_b y = a^n xe_b y = a^n e_a xe_b y.$$

Multiplying both sides of this equality by $(a^n)^{-1}$ from the left we obtain

$$e_a a = (a^n)^{-1} a^{n+1} = e_a xe_b y.$$

Since $e_a a = ae_a$ by Corollary 1 it follows that $ae_a = e_a xe_b y$. Similarly, $be_b = e_b ye_a x$. Therefore, $ae_a \sim_p be_b$. $\square$

Lemma 7. *Let $a, b \in S$ be group elements satisfying $a \mathcal{H} b$ and $a \sim_p b$. Then there exists $h \in H_a$ such that $a = h^{-1}bh$.*

Proof. By Lemma 5 $a = hg$, $b = gh$ for some $h, g \in H_a$. Thus, $ah^{-1} = h^{-1}b$, which implies $a = h^{-1}bh$ as required. $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. (1). Necessity. Let $a \sim_p b$. Fix a pair of mutually inverse elements $t \in R_a \cap L_b$ and $t' \in R_b \cap L_a$ (this is possible to do by Proposition 2). In particular, $e_a = tt'$, $e_b = t't$. Fix also some $x \in R_a \cap L_b$ and $y \in R_b \cap L_a$ such that $a = xy$ and $b = yx$ (such elements exist by Lemma 5). Then $b \mathcal{H} t'at = t'x \cdot yt$ and $yt \cdot t'x = ye_b x = yx = b$. It follows that $t'at \sim_p b$ and $t'at \mathcal{H} b$. Now Lemma 7 ensures us that there is $g \in H_b$ such that $b = g^{-1}t'atg$. Set $u = g^{-1}t'$, $v = tg$. Since $L_g \cap R_{t'} = H_g$ contains an idempotent then $u = g^{-1}t' \in R_g \cap L_{t'} = H_{t'}$. Similarly, $v \in H_t$. Furthermore, $uvu = g^{-1}t'tgg^{-1}t' = u$, $vuv = tgg^{-1}t'tg = v$. Thereby, using Proposition 2, we have that $\rho_u \circ \lambda_v$ is an isomorphism from H_a to H_b. It remains to note, that $b = uav$ and $a = vbu$.

Sufficiency. Suppose $b = uav$ and $a = vbu$, where u, v are mutually inverse. Then $b = uvbuv$ which implies that $uvb = uvbuv$. The two previous equalities imply $b = uvbuv = uvb$. Denote $s = vb$, $t = u$. Then $st = a$ and $ts = uvb = b$. Hence, $a \sim_p b$.

(2). Clearly, we have to show only that $a \sim b$ implies $a \sim_p b$. Suppose that a and b are conjugate in at most n steps and that $a = a_0, a_1, \ldots, a_n = b$ such that $a_i \sim_p a_{i+1}$, $0 \leq i \leq n - 1$ are fixed. Corollary 2 implies that all a_i are group-bound. Apply induction on n. If $n = 1$ there is nothing to prove.

Let $n = 2$. Suppose $a \sim_p a_1$, $a_1 \sim_p b$. By the first statement of this Theorem $a = ta_1s$, where $ts = e_a$, $st = e_{a_1}$, and $a_1 = ubv$, where $uv = e_{a_1}$, $vu = e_b$. Then $a = tubvs$, $b = va_1u = vsatu$ and $tuvs = te_{a_1}s = ts = e_a$, $vstu = ve_{a_1}u = vu = e_b$.

Let $n \geq 2$. Assume that any two group elements, which are conjugate in at most k steps with $k \leq n - 1$, are primarily conjugate. It follows from Lemma 6 that

$$a = a_0e_{a_0} \sim_p a_1e_{a_1} \sim_p \cdots \sim_p a_ne_{a_n} = b.$$

Note that all $a_ie_{a_i}$ are group elements by Corollary 1. Then $a \sim_p ae_{a_{n-1}}$ by the inductive assumption. It follows that $a \sim_p a_{n-1}e_{a_{n-1}} \sim_p b$. The inductive assumption implies now that $a \sim_p b$. $\square$

The following statements are direct consequences of Theorem 1.

Corollary 3. *Let S be a semigroup. Suppose $a, b \in S$ are group elements. Then $a \sim b$ implies $a\mathcal{D}b$.*

Corollary 4. *Let S be a completely regular semigroup. Then the relations $\sim_p$ and $\sim$ on S coincide. In particular, $\sim_p$ is an equivalence relation.*

Corollary 5. *Let S be a band. Then $a \sim b$ if and only if $a\mathcal{D}b$.*

4   The general case

To prove the results of this Section we will use the results of the previous Section and one important observation, from which we start.

Proposition 3. *Let S be a regular semigroup and $a \in S$ a group-bound element. Then $a \sim ae_a$.*

Proof. Let t be the height of a (see the definition in the preliminaries). For the case $t = 1$ the statement is obvious as $ae_a = a$. Suppose $t \geq 2$. For each i, $1 \leq i \leq t - 1$, denote by α_i any element which is inverse of a^i, and by α_t the element $(a^t)^{-1}$, which is inverse of a^t in the group H_{a^t}. Put $c_0 = a$, $c_i = a^{i+1}\alpha_i$, $1 \leq i \leq t$. Note that for $0 \leq i \leq t - 1$ we have

$$a^i\alpha_i \cdot a^{i+1}\alpha_{i+1} = a^i\alpha_i a^i \cdot a\alpha_{i+1} = a^{i+1}\alpha_{i+1}. \tag{1}$$

Let $s = c_i$, $t = a^{i+1}\alpha_{i+1}$. Then using (1) we obtain

$$st = a \cdot a^i\alpha_i \cdot a^{i+1}\alpha_{i+1} = a \cdot a^{i+1}\alpha_{i+1} = c_{i+1};$$

$$ts = a^{i+1}\alpha_{i+1} \cdot a^{i+1}\alpha_i = a \cdot a^i\alpha_i = c_i.$$

It follows that $c_i \sim_p c_{i+1}$, $0 \leq i \leq t - 1$. Therefore, $a = c_0 \sim c_t = ae_a$. $\quad\square$

Corollary 6. *Let the semigroup S be regular and $a, b \in S$ be group-bound elements. Then $a \sim b$ if and only if $ae_a \sim be_b$.*

Theorem 2. *Let S be a regular epigroup and $a, b \in S$. Then $a \sim b$ if and only if there exists a pair of mutually inverse elements $u, v \in S$ such that $ae_a = u \cdot be_b \cdot v$ and $be_b = v \cdot ae_a \cdot u$.*

Proof. The statement follows from Corollary 1, Theorem 1 and Corollary 6. $\quad\square$

Corollary 7. *Let S be a regular semigroup with the zero element 0. Then any two nilpotent elements are conjugate, and if $a \sim b$ and a is nilpotent then b is also nilpotent.*

Proof. Suppose that a, b are nilpotent. Then $ae_a = be_b = 0$, so that $a \sim b$ by Theorem 2.

Suppose now that a is nilpotent and $a \sim b$. This and Corollary 6 imply that $be_b \sim ae_a = 0$. It follows now from Corollary 3 that $be_b \mathcal{D} 0$. Thus $be_b = 0$, and hence $e_b \mathcal{H} be_b = 0$, so that $e_b = 0$. Therefore, b is nilpotent. $\square$

Recall (see [5], p.199) that an inverse semigroup S with the group of units G is called *factorizable* provided that for each $s \in S$ there is $g \in G$ such that $s \leq g$ with respect to the natural partial order on S i.e. $ss^{-1} = sg^{-1}$.

The following theorem provides a characterization of conjugacy in terms of G-conjugacy for the class of factorizable inverse epigroups.

Theorem 3. *Let S be a factorizable inverse epigroup with the identity element e and the group of units G. Let $a, b \in S$. Then $a \sim b$ if and only if $ae_a \sim_G be_b$.*

Proof. Since $\sim_G \subset \sim$ and in view of Corollary 6 it is enough to prove only that $a \sim b$ implies $ae_a \sim_G be_b$. Suppose $a \sim b$. It follows from Theorem 2 that $ae_a = sbe_b t$ and $be_b = tae_a s$ for some mutually inverse $s \in R_a \cap L_b$ and $t \in R_b \cap L_a$. Since S is an inverse semigroup it follows that t coincides with s^{-1} — the (unique) element, inverse to s. That $s^{-1}s\mathcal{H}b$ yields $s^{-1}sbs^{-1}s = b$. Let $g \in G$ be such that $s \leq g$. Then $a = sbs^{-1} = ss^{-1}sbs^{-1}ss^{-1} = gs^{-1}sbs^{-1}sg^{-1} = gbg^{-1}$, and the proof is complete. $\square$

5 Some examples

5.1 Finite transformation semigroups $\mathcal{IS}_n$, $\mathcal{T}_n$ and $\mathcal{PT}_n$

Let $\mathcal{IS}_n$ be the *full finite inverse symmetric semigroup*, i.e. the semigroup of all partial permutations over an n-element set $X = \{1, \ldots, n\}$. The group of units of $\mathcal{IS}_n$ is the full symmetric group $\mathcal{S}_n$ of all everywhere defined permutations. The idempotents of $\mathcal{IS}_n$ are precisely the identity maps on subsets of X. Let $\pi \in \mathcal{IS}_n$. Set G_π to be the directed graph whose set of vertices $V(G_\pi)$ coincides with X, and $(x, y) \in E(G_\pi)$ if and only if $\pi(x) = y$. The graph G_π is called the *graph of action* of π. There are two types of connected components of G_π: *cycles and chains* (see [9, 2, 6]). The *cyclic type* and the *chain type* of π are respectively the (unordered) tuples $(n_1, \ldots, n_k)$ and $(l_1, \ldots, l_t)$, where $n_1, \ldots, n_k$ are the lengthes of the cycles of π, and $l_1, \ldots, l_t$ are the lengthes of the chains of π (by the length of a chain $a_1 \to a_2 \to \cdots \to a_n \to \varnothing$ we mean here the number n of its vertices). The following Lemma is straightforward.

Lemma 8. *1. $\pi \in \mathcal{IS}_n$ is a group element if and only if all its chains are trivial, i.e. of length 1.*

2. *If the cyclic and chain types of π are respectively $(n_1, \ldots, n_k)$ and $(l_1, \ldots, l_t)$ then the cyclic and chain types of πe_π are respectively $(n_1, \ldots, n_k)$ and $(1, \ldots, 1)$.*

3. *$\pi \sim_{S_n} \tau$ if and only if the graphs G_π and G_τ are isomorphic as directed graphs, which is the case if and only if the cyclic and chain types of π and τ coincide.*

Since $\mathcal{IS}_n$ is factorizable, Theorem 3 is applicable. Together with Lemma 8 it gives the following criterion of conjugacy for the semigroup $\mathcal{IS}_n$.

Theorem 4 ([9, 2]). *Let $\pi, \tau \in \mathcal{IS}_n$. Then $\pi \sim \tau$ if and only if the cyclic types of π and τ coincide.*

Let $\mathcal{T}_n$ and $\mathcal{PT}_n$ be the the semigroups of respectively all transformations and of all partial transformations (in both cases not necessarily injective) of the set $X = \{1, \ldots, n\}$. Both of these semigroups are regular while not inverse. In the same vein as it was done in the case of $\mathcal{IS}_n$ we define the graph of action G_π for $\pi \in \mathcal{T}_n$ or $\pi \in \mathcal{PT}_n$. Let $\pi \in \mathcal{T}_n$ or $\pi \in \mathcal{PT}_n$. One can easily make sure that each connected component of G_π contains no more than one cycle. By the cyclic type of π we will mean the (unordered) tuple $(n_1, \ldots, n_k)$, where $n_1 \ldots, n_k$ are the lengthes of cycles of π. Denote the range of π by $\mathrm{ran}\pi$, and the kernel of π by $\ker\pi$. Recall that the kernel of π is such a partition of the domain of π that $a, b \in X$ belong to the same block if and only if $a\pi = b\pi$.

Lemma 9. *Let $\pi \in \mathcal{T}_n$ or $\pi \in \mathcal{PT}_n$. Then*

1. *The cyclic types of π and πe_π coincide.*

2. *π is a group element if and only if $\mathrm{ran}\pi$ is a transversal of $\ker\pi$. In the latest case the restriction $\overline{\pi}$ of π to $\mathrm{ran}\pi$ is a permutation on the set $\mathrm{ran}\pi$ and is a group element of $\mathcal{IS}_n$.*

3. *If π is a group element then the cyclic types of π and $\overline{\pi}$ coincide.*

4. *Two group elements $\pi, \tau \in \mathcal{T}_n$ (or $\mathcal{PT}_n$) are conjugate if and only if $\overline{\pi}$ and $\overline{\tau}$ are conjugate in $\mathcal{IS}_n$.*

Proof. 1. Let $\mathrm{stran}\pi = \cap_{k \geq 1}\mathrm{ran}\pi^k$ be the stable range of π. For $a \in X$ we have that $a \in \mathrm{stran}\pi$ if and only if a belongs to a cycle in G_π. This and that e_π acts identically on $\mathrm{stran}\pi$ imply that π and πe_π have the same cycles (see also [6]).

2. Follows from the description of Green's relations in $\mathcal{T}_n$ and $\mathcal{PT}_n$ (see, for example, [4]).

3. Since $\operatorname{ran}\pi = \operatorname{stran}\pi$ in the case of the group element π, it follows that the graph of action $G_{\overline{\pi}}$ of $\overline{\pi}$ is the union of cycles of G_π, whence the cyclic types of π and $\overline{\pi}$ coincide.

4. Let first $\pi \sim \tau$. It follows from Theorem 2 and its proof that there are mutually inverse elements $t \in R_\pi \cap L_\tau$ and $t' \in R_\tau \cap R_\pi$ such that $\pi = t\tau t'$ and $\tau = t'\pi t$. It follows from the description of Green's relations on $\mathcal{T}_n$ and $\mathcal{PT}_n$ that

$$\ker t = \ker\tau, \ \ \ker t' = \ker\pi, \ \ \operatorname{ran} t = \operatorname{ran}\pi, \ \ \operatorname{ran} t' = \operatorname{ran}\tau.$$

Let $\overline{t}$ be the restriction of t to $\operatorname{ran}\tau$ and $\overline{t}'$ — the restriction of t' to $\operatorname{ran}\pi$. It follows that $\overline{t}\,\overline{t}'$ is the identity map on $\operatorname{ran}\pi$ and $\overline{t}'\overline{t}$ is the identity map on $\operatorname{ran}\tau$. Hence, $\overline{t}$ and $\overline{t}'$ are the pair of mutually inverse elements from $\mathcal{IS}_n$. This, $\overline{\pi} = \overline{t}\,\overline{\tau}\,\overline{t}'$, $\overline{\tau} = \overline{t}'\overline{\pi}\overline{t}$ and Theorem 2 imply that $\overline{\pi}$ and $\overline{\tau}$ are $\mathcal{IS}_n$-conjugate.

Now let the partial permutations $\overline{\pi}$ and $\overline{\tau}$ be $\mathcal{IS}_n$-conjugate. Then there exist $\overline{t} \in R_{\overline{\pi}} \cap L_{\overline{\tau}}$ and $\overline{t}' \in R_{\overline{\tau}} \cap R_{\overline{\pi}}$ such that $\overline{\pi} = \overline{t}\,\overline{\tau}\,\overline{t}'$ and $\overline{\tau} = \overline{t}'\overline{\pi}\overline{t}$. Define the elements $t, t' \in \mathcal{T}_n$ ($\mathcal{PT}_n$) as follows. Set t to be such that $\ker t = \ker\tau$, $\operatorname{ran} t = \operatorname{ran}\pi = \operatorname{ran}\overline{t}$ and the restriction of t to $\operatorname{ran}\tau$ coincides with $\overline{t}$. Similarly, set t' to be such that $\ker t' = \ker\pi$, $\operatorname{ran} t' = \operatorname{ran}\tau = \operatorname{ran}\overline{t}'$ and the restriction of t' to $\operatorname{ran}\pi$ coincides with $\overline{t}'$. Note that it follows from the definitions of $\overline{t}$ and $\overline{t}'$ that t and t' can be constructed uniquely and happen to be mutually inverse. Moreover, the construction of t and t' implies $\pi = t\tau t'$ and $\tau = t'\pi t$. Then by Theorem 2 π and τ are conjugate. $\qquad\square$

As a corollary we obtain the criterion of $\mathcal{T}_n$- (or $\mathcal{PT}_n$-) conjugacy in terms of cyclic types of elements.

Theorem 5 ([6]). *Let $\pi, \tau \in \mathcal{T}_n$ ($\mathcal{PT}_n$). Then π and τ are $\mathcal{T}_n$- ($\mathcal{PT}_n$-) conjugate if and only if their cyclic types coincide.*

5.2 Full semigroups of linear transformations of a finitely dimensional vector space

Let F be a field and V_n be an n-dimensional vector space over F. An isomorphism $\varphi : U \to W$, where U, W are some subspaces of V_n is called a *partial automorphism* of V_n with the domain $\operatorname{dom}\varphi = U$ and the range $\operatorname{ran}\varphi = W$. The set of partial automorphisms of V_n with respect to the composition of partial automorphisms is an inverse semigroup and denoted by $\operatorname{PAut}(V_n)$. Let $\varphi \in \operatorname{PAut}(V_n)$. For each positive integer k we have the inclusions

$$\operatorname{dom}\varphi \supset \operatorname{dom}\varphi^k \supset \operatorname{dom}\varphi^{k+1} \supset \{0\},$$

implying that

$$n \geq \dim U \geq \dim(\mathrm{dom}\varphi^2) \geq \cdots \geq \dim(\mathrm{dom}\varphi^k) \geq \cdots \geq 0.$$

Since at most n of this inequalities are strict we can assert that starting from some power t we have $\mathrm{dom}\varphi^t = \mathrm{dom}\varphi^{t+i}$ for each $i \geq 0$. It follows that $\mathrm{dom}\varphi^t = \mathrm{ran}\varphi^t$, so that $\varphi^t \in \mathrm{GL}(\mathrm{dom}\varphi^t)$ is a group element of $\mathrm{PAut}(V_n)$, which shows that $\mathrm{PAut}(V_n)$ is an epigroup (the notation $\mathrm{GL}(W)$ stands for the full linear group over the subspace W). It is easily proved that $\mathrm{PAut}(V_n)$ is factorizable. From Theorem 3 we derive the following criterion of $\mathrm{PAut}(V_n)$-conjugacy.

Theorem 6. *Let $\varphi, \psi \in \mathrm{PAut}(V_n)$. Then φ and ψ are $\mathrm{PAut}(V_n)$-conjugate if and only if φe_φ and ψe_ψ are $\mathrm{GL}(V_n)$-conjugate.*

Now switch to the regular semigroups $\mathrm{End}(V_n)$ and $\mathrm{PEnd}(V_n)$ of respectively all endomorphisms and all partial endomorphisms of V_n. The same arguments as in the case of $\mathrm{PAut}(V_n)$ show that both $\mathrm{End}(V_n)$ and $\mathrm{PEnd}(V_n)$ are epigroups.

Lemma 10. *Let S denote one of the semigroups $\mathrm{End}(V_n)$ or $\mathrm{PEnd}(V_n)$.*

1. *$\pi \in S$ is a group element if and only if $\mathrm{dom}\pi$ decomposes into the direct sum $\mathrm{dom}\pi = \mathrm{ran}\pi \oplus \ker\pi$. In the latest case the restriction $\overline{\pi}$ of π to $\mathrm{ran}\pi$ is an automorphism of $\mathrm{ran}\pi$ which is a group element of $\mathrm{PAut}(V_n)$.*

2. *Two group elements $\pi, \tau \in S$ are S-conjugate if and only if $\overline{\pi}$ and $\overline{\tau}$ are $\mathrm{PAut}(V_n)$- conjugate.*

Proof. 1. Recall that $\pi \in S$ is a group element if and only if H_π contains an idempotent, i.e. some projection map $e = e(V_1, V_2)$, such that $\mathrm{dom}\,e = V_1 \oplus V_2$ and e is a projecting of $\mathrm{dom}\,e$ onto V_1 parallelly to V_2. The statement now follows from the fact that $\pi \mathcal{H} e$ if and only if $\ker\pi = \ker e$ and $\mathrm{ran}\pi = \mathrm{ran}e$.
2. The proof is similar to the proof of the fourth statement of Lemma 9. $\square$

As a corollary we obtain the criterion of conjugacy (where $S = \mathrm{End}(V_n)$ or $S = \mathrm{PEnd}(V_n)$) in terms of G-conjugacy.

Theorem 7 ([7] for the case of $\mathrm{End}(V_n)$). *Let S denote one of the semigroups $\mathrm{End}(V_n)$ or $\mathrm{PEnd}(V_n)$ and $\varphi, \psi \in S$. Then φ and ψ are S-conjugate if and only if $\overline{\varphi_e\varphi}$ and $\overline{\psi e_\psi}$ are $\mathrm{GL}(V_n)$-conjugate.*

11

5.3 Partial automatic permutations over a finite alphabet

Recall that a *Mealy automaton over a finite alphabet* X is a triple $\mathcal{A} = (X, \varphi, \psi)$, where Q is the set of *internal states* of the automaton, $\varphi : Q \times X \to Q$— its *transition function* and $\psi : Q \times X \to X$ – its *output function*. In the case when the functions φ and ψ are everywhere defined the automaton $\mathcal{A}$ is called *full*, otherwise it is called *partial*. An automaton $\mathcal{A}$ is called *initial* if a state $q_o \in Q$ is marked as an *initial state*. Each initial automaton $(\mathcal{A}, q_0)$ over X defines a (partial) transformation of the set X^* of all words over X by extending functions φ and ψ to the set $Q \times X^*$ as follows:

$$\varphi(q, e) = q, \quad \varphi(q, wx) = \varphi(\varphi(q, w), x);$$

$$\psi(q, e) = e, \quad \psi(q, wx) = \psi(\varphi(q, w), x),$$

where e denotes the empty word. Now define the transformation $f_{\mathcal{A}, q_0} : X^* \to X^*$ via

$$f_{\mathcal{A}, q_0}(u) = \psi(q_0, x_1)\psi(\varphi(q_0, x_1), x_2)\psi(\varphi(q_0, x_1 x_2), x_3)..., \tag{2}$$

where $u = x_1 x_2 x_3 \cdots \in X^*$. The expression in the right-hand side of (2) is undefined if and only if at least one of the values of φ or ψ in it is undefined. Partial injective transformations which is defined by some partial initial automaton is called a *partial automatic permutation* (or, in other terminology, a *letter-to-letter transduction*). The set of all partial automatic permutations over X with respect the composition of maps is an inverse semigroup which will be denoted by $\mathcal{ISA}(X)$. Note that a partial automatic permutation is a group element of $\mathcal{ISA}(X)$ if and only if its graph of action has no chains of length greater than one. The following Lemma is straightforward.

Lemma 11. *An element $f \in \mathcal{ISA}(X)$ is group-bound if and only if the lengthes of its chains are uniformly bounded.*

A partial automatic permutation $g \in \mathcal{ISA}(X)$ is said to be *finitary* if there exists $l \geq 0$ such that for every word $x_1 x_2 \cdots \in X^*$ belonging to the domain of g and its image $y_1 y_2 \cdots = (x_1 x_2 \ldots)^g$ one has $x_i = y_i$ for all $i \geq l$. The set $Fin\mathcal{ISA}(X)$ of all finitary partial automatic permutations is an inverse subsemigroup of $\mathcal{ISA}(X)$ and by Lemma 11 it is an epigroup. The group of units of $Fin\mathcal{ISA}(X)$ coincides with the group $Fin\mathcal{SA}(X)$ consisting of all everywhere defined elements of $Fin\mathcal{ISA}(X)$. It is easily seen that the semigroup $Fin\mathcal{SA}(X)$ is factorizable. From Theorem 3 we obtain the following conjugacy criterion for this semigroup.

Theorem 8. *Two elements $f, g \in Fin\mathcal{ISA}(X)$ are conjugate with respect to $\sim$ if and only if fe_f and ge_g are $Fin\mathcal{SA}(X)$-conjugate.*

Theorem 2 also gives us criteria of conjugacy for the regular epigroups $Fin\mathcal{A}(X)$ of all (not necessarily injective) *finitary automatic transformations of X^** and $Fin\mathcal{PA}(X)$ of all *partial finitary automatic transformations of X^**. It easily seen that a statement similar to Lemma 9 holds for these semigroups. Therefore we obtain the following conjugacy criterion.

Theorem 9. *Let S denote one of the semigroups $Fin\mathcal{A}(X)$ or $Fin\mathcal{PA}(X)$. Two elements $f, g \in S$ are S-conjugate if and only if $\overline{fe_f}$ and $\overline{ge_g}$ are $Fin\mathcal{SA}(X)$-conjugate.*

Appendix A

Here we are going to show that for $|X| \geq 2$ the semigroup $\mathcal{ISA}(X)$ is not an epigroup. For this we give an example of an automaton $(\mathcal{A}, q_0)$ with four states over a two-letter alphabet $X = \{0, 1\}$ such that $f_{\mathcal{A}, q_0}$ is not a group-bound element of $\mathcal{ISA}(X)$. That finite automata $(\mathcal{A}, q_0)$ such that $f_{\mathcal{A}, q_0}$ is not group-bound exist is rather evident. However, so far as to our knowledge, this fact has never been indicated in the literature. Besides, from Corollary 2 it follows that a non group-bound element can not be conjugate to a group element. This and the existence of non group-bound elements assure that the conjugacy criterion for the semigroup $\mathcal{ISA}(X)$ announced in [10] is incorrect.

Construct $(\mathcal{A}, q_0) = (Q, \varphi, \psi, q_0)$ as follows. Let $Q = \{A, B, C, D\}$, $q_0 = A$ and

$$\varphi(A, 0) = D, \quad \varphi(A, 1) = B \quad \varphi(B, 0) \text{ undefined}, \quad \varphi(B, 1) = C,$$

$$\varphi(C, 0) = C, \quad \varphi(C, 1) = C, \quad \varphi(D, 0) = A, \quad \varphi(D, 1) \text{ undefined};$$

$$\psi(A, 0) = 1, \quad \psi(A, 1) = 0, \quad \psi(B, 0) \text{ undefined}, \quad \psi(B, 1) = 0,$$

$$\psi(C, 0) = 0, \quad \psi(C, 1) = 1, \quad \psi(D, 0) = 1, \quad \psi(D, 1) \text{ undefined}.$$

The Moore diagram of the constructed automaton is given in Figure 1, where the initial state is marked by a double circle, and there is no arrow with the first label $x \in X$ beginning in a state $q \in Q$ if and only if $\varphi(q, x)$ and $\psi(q, x)$ are undefined.

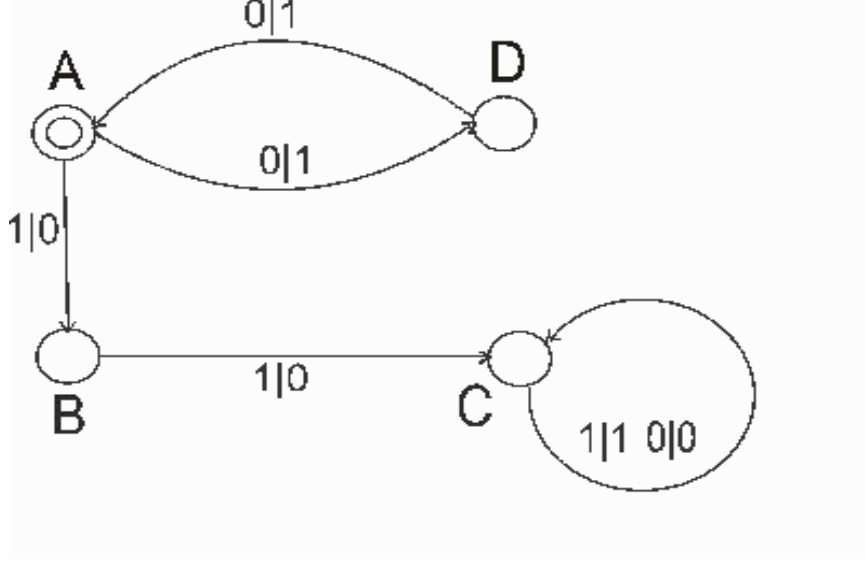**Lemma 12.** *$f_{\mathcal{A}, q_0}$ is not a group-bound element of $\mathcal{ISA}(\{0, 1\})$.*

Figure 1:

Proof. Show first that for any $k \geq 1$ the orbit (with respect to the action of $f_{\mathcal{A},q_0}$) of the word $\underbrace{1\dots1}_{2k}$ is a cycle of length 2^k. Apply induction on k. For $k = 1$ we have $11 \mapsto 00 \mapsto 11$. Suppose that $k \geq 1$ and

$$\underbrace{1\dots1}_{2k} = u_1 \mapsto u_2 \mapsto \dots \mapsto u_{2^k} = \underbrace{0\dots0}_{2k} \mapsto u_1.$$

It follows from the definition of $(\mathcal{A}, q_0)$ that

$$\underbrace{1\dots1}_{2k+2} = u_1 11 \mapsto u_2 11 \mapsto \dots \mapsto u_{2^k} 11 = \underbrace{0\dots0}_{2k} 11 \mapsto$$

$$\underbrace{1\dots1}_{2k} 00 = u_1 00 \mapsto u_2 00 \mapsto \dots \mapsto u_{2^k} 00 = \underbrace{1\dots1}_{2k+2} \mapsto u_1 11,$$

as required. Let now $k \geq 1$. Then

$$\underbrace{1\dots1}_{2k} 01 = u_1 01 \mapsto u_2 01 \mapsto \dots \mapsto u_{2^k} 01 = \underbrace{0\dots0}_{2k} 01,$$

and $f_{\mathcal{A},q_0}(\underbrace{0\dots0}_{2k} 01)$ is undefined. Therefore, the word $\underbrace{1\dots1}_{2k} 01$ belongs to the chain of length at least 2^k, $k \geq 1$. The statement now follows from Lemma 11. $\qquad\square$

Corollary 8. *If $|X| \geq 2$ then in $\mathcal{ISA}(X)$ there are conjugacy classes without group elements.*

Proof. This follows from Corollary 2 and Lemma 12. $\qquad\square$

14

References

[1] C. Choffrut. Conjugacy in free inverse monoids. Internat. J. Algebra Comput. 3 (1993), no. 2, 169-188.

[2] O.G. Ganyushkin, T.V. Kormysheva. The chain decomposition of partial permutations and classes of conjugate elements of the semogroup $\mathcal{IS}_n$. Visnyk of Kyiv University, 2 (1993), 10-18 (in Ukrainian).

[3] P.W. Gawron, V.V. Nekrashevych, V.I. Sushchansky. Conjugation in tree automorphism groups, Internat. J. Algebra Comput. 11 (2001), no. 5, 529-547.

[4] P.M. Higgins. Techniques of semigroup theory. Oxford University Press, 1992.

[5] J.M. Howie. Fundamentals of Semigroup Theory. Clarendon Press, Oxford, 1995.

[6] G. Kudryavtseva, V. Mazorchuk. On conjugation in some transformation and Brauer-type semigroups, to appear in Math. Publ. Debrecen.

[7] G. Kudryavtseva, V. Mazorchuk. Square matrices as a semigroup, Preprint, Uppsala University, 2003.

[8] G. Lallement. Semigroups and combinatorial applications, John Wiley &Sons, New York, 1979.

[9] S. Lipscomb. Symetric inverse semigroups. Mathematical Surveys and monographs, 46, Providence, RI, 1996.

[10] A.S. Olijnyk, V.I. Sushchansky. Conjugacy in the inverse semigroups of partial permutations over a finite alphabet, Proc. of the Ukrainian Academy of Sciences, vol. 9 (2004), 35-39 (in Russian).

[11] L.N. Shevrin. On the theory of epigroups I, Russian Acad. Sci. Math. Sb. 185 (1994), 8, 129-160 (in Russian).